

AN ANALYTIC GROTHENDIECK RIEMANN ROCH THEOREM

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ABSTRACT. We extend the Boutet de Monvel Toeplitz index theorem to complex manifold with isolated singularities following the relative K -homology theory of Baum, Douglas, and Taylor for manifold with boundary. We apply this index theorem to study the Arveson-Douglas conjecture. Let \mathbb{B}^m be the unit ball in \mathbb{C}^m , and I an ideal in the polynomial algebra $\mathbb{C}[z_1, \dots, z_m]$. We prove that when the zero variety Z_I is a complete intersection space with only isolated singularities and intersects with the unit sphere \mathbb{S}^{2m-1} transversely, the representations of $\mathbb{C}[z_1, \dots, z_m]$ on the closure of I in $L_a^2(\mathbb{B}^m)$ and also the corresponding quotient space Q_I are essentially normal. Furthermore, we prove an index theorem for Toeplitz operators on Q_I by showing that the representation of $\mathbb{C}[z_1, \dots, z_m]$ on the quotient space Q_I gives the fundamental class of the boundary $Z_I \cap \mathbb{S}^{2m-1}$. In the appendix, we prove with Kai Wang that if $f \in L_a^2(\mathbb{B}^m)$ vanishes on $Z_I \cap \mathbb{B}^m$, then f is contained inside the closure of the ideal I in $L_a^2(\mathbb{B}^m)$.

1. INTRODUCTION

Let X and Y be closed smooth complex manifolds, and $f : X \rightarrow Y$ be a proper smooth map. The classical Grothendieck-Riemann-Roch theorem [8] relates the push forward maps on the Grothendieck groups $f_! : K_0(X) \rightarrow K_0(Y)$ of bounded complexes of coherent sheaves and the Chow groups $f_* : A(X) \rightarrow A(Y)$ of subvarieties modulo rational equivalence. More precisely, let $\text{Ch} : K_0(X) \rightarrow A(X)$ be the Chern character map, and $\text{Td}(X) \in A(X)$ be the Todd genus of X . The Grothendieck-Riemann-Roch theorem states that for a vector bundle E on X ,

$$\text{Ch}(f_!(E)) \cup \text{Td}(Y) = f_*(\text{Ch}(E) \cup \text{Td}(X)).$$

In noncommutative geometry, the push forward map $f_* : K_\bullet(X) \rightarrow K_\bullet(Y)$ on the K -homology group introduced by Brown, Douglas, and Fillmore [12] is related to the push forward map $f_* : HP^\bullet(C^\infty(X)) \rightarrow HP^\bullet(C^\infty(Y))$ on the periodic cyclic cohomology introduced by Connes [15] via the Connes-Chern character $\text{Ch} : K_\bullet(-) \rightarrow HP^\bullet(-)$ satisfying

$$\text{Ch}(f_*([D])) = f_*(\text{Ch}([D])).$$

This can be viewed as a noncommutative generalization of the Grothendieck-Riemann-Roch theorem.

In this article, motivated from questions in operator theory, we are interested in extending the above study of push forward maps in two directions.

- (1) Allow X and Y to have singularities.
- (2) Allow X and Y to have boundaries.

In the literature, many interesting works have been developed to address the above questions. For example, Baum, Fulton, and McPherson in [5], [6] proved the Riemann-Roch theorem for a singular projective variety X ; Baum and the first author in [3], [4] studied the relative K -homology groups, $K_\bullet(X, \partial X)$, for a manifold X with boundary, ∂X . However, the formulation of the general Grothendieck-Riemann-Roch theorem, including the above two special cases, is missing.

In this paper, we study the generalization of the Grothendieck-Riemann-Roch theorem in the following setting. Let $A = \mathbb{C}[z_1, \dots, z_m]$ be the polynomial ring of m variables. Let I be an ideal of A . Define

$$Z_I = \{(z_1, \dots, z_m) \in \mathbb{C}^m : a(z_1, \dots, z_m) = 0, \forall a \in I\}.$$

We point out that the analytic space Z_I may have singularities. Let \mathbb{B}^m be the open unit ball in \mathbb{C}^m with the natural euclidean metric, and $\partial\mathbb{B}^m := \overline{\mathbb{B}^m} \setminus \mathbb{B}^m$ be its boundary, the unit sphere \mathbb{S}^{2m-1} . Denote $Z_I \cap \mathbb{B}^m$ by Ω_I . The analytic space Ω_I is naturally a (singular) submanifold of \mathbb{B}^m with the boundary $\partial\Omega_I := \overline{\Omega_I} \setminus \Omega_I = Z_I \cap \partial\mathbb{B}^m$.

When Z_I is smooth and intersects the unit sphere $\mathbb{S}^{2m-1} = \partial\mathbb{B}^m$ transversely, $\overline{\Omega_I}$ is a smooth complex manifold with a strongly pseudoconvex boundary $\partial\Omega_I = Z_I \cap \mathbb{S}^m$ (of odd dimension). Baum, Douglas, and Taylor in [4] introduced a K -homology class $[D_N]$ in $K_0(\overline{\Omega_I}, \partial\Omega_I)$ from the $\bar{\partial}$ -operator on the Dolbeault complex of Ω_I with the Neumann boundary condition. And they proved that the boundary map $\partial : K_0(\overline{\Omega_I}, \partial\Omega_I) \rightarrow K_1(\partial\Omega_I)$ maps $[D_N]$ to the fundamental class on $\partial\Omega_I$ naturally associated to the CR -structure on $\partial\Omega_I$. Furthermore, the fundamental class on $\partial\Omega_I$ is the spin^c Dirac operator on $\partial\Omega_I$ associated to the CR -structure. This identification of $\partial([D_N])$ provides a different proof of Boutet de Monvel's Toeplitz index theorem [9].

In this article, we extend the above Baum-Douglas-Taylor result to the following case. Let I be generated by $p_1, \dots, p_M \in A = \mathbb{C}[z_1, \dots, z_m]$ with $M \leq m - 2$. We make the following assumptions.

- Assumption 1.1.**
- (1) The Jacobian matrix $(\partial p_i / \partial z_j)_{i,j}$ is of maximal rank on the boundary $\partial\Omega_I = Z_I \cap \partial\mathbb{B}^m$;
 - (2) Z_I intersects $\partial\mathbb{B}^m$ transversely.

Under Assumption 1.1, Ω_I is an analytic space of complex dimension $k := m - M \geq 2$ and complex codimension M . Ω_I has a smooth strongly pseudoconvex boundary $\partial\Omega_I = Z_I \cap \partial\mathbb{B}^m$ and (possibly) a finite number of isolated singularities away from the boundary. Furthermore, by the assumption on the Jacobian matrix, Ω_I is a complete intersection space [19, Sec.18.5], from which we can conclude that the ideal $I \subset A$ is radical.

Let Σ_I denote the set of singular points of Ω_I . The space $\Omega_I^0 := \Omega_I - \Sigma_I$ is a smooth submanifold of \mathbb{B}^m and inherits a natural riemannian metric σ_I from the one on \mathbb{B}^m . Let dV_I be the volume element on Ω_I^0 defined by this metric σ_I . Consider the operator $D_N = \bar{\partial} + \bar{\partial}^*$ on the Dolbeault complex of Ω_I^0 with the Neumann boundary condition.

As $\partial\Omega_I$ is smooth and strongly pseudoconvex, the restriction of the complex structure to the boundary defines a CR -structure and therefore a spin^c structure on $\partial\Omega_I$. The Dirac operator $D_{\partial\Omega_I}$ associated to this spin^c -structure is a fundamental class of $K_1(\partial\Omega_I)$. The following theorem generalizes the results in [4].

Theorem 1.2. *Under Assumption 1.1, the operator D_N gives a relative K -homology class in $K_0(\overline{\Omega}_I, \partial\Omega_I)$. And the boundary map $\partial : K_0(\overline{\Omega}_I, \partial\Omega_I)$ maps $[D_N]$ to the fundamental class on $\partial\Omega_I$ which is associated to the CR -structure on $\partial\Omega_I$.*

We remark that Theorem 1.2 holds true in much more general cases than Assumption 1.1. We refer the readers to Remark 3.3 for the more precise statement.

Theorem 1.2 can be viewed as an analytic version of the Grothendieck-Riemann-Roch theorem. When I is a homogeneous ideal of $A = \mathbb{C}[z_1, \dots, z_m]$, the zero variety Z_I is invariant under the multiplication by $\mathbb{C}^* = \mathbb{C} - \{0\}$. Assume that the origin is the only possible isolated singular point of Z_I . Z_I intersects with the unit sphere \mathbb{S}^{2m-1} transversely, and therefore the boundary $\partial\Omega_I$ is a smooth submanifold of \mathbb{S}^{2m-1} . The group S^1 as the unit circle in \mathbb{C}^* acts on the unit sphere \mathbb{S}^{2m-1} freely. As I is a homogeneous ideal, $\partial\Omega_I \subset \mathbb{S}^{2m-1}$ is invariant under the S^1 -action. Furthermore the class D_N (and $\partial[D_N]$) is S^1 -equivariant and lives in $K_0^{S^1}(\overline{\Omega}_I, \partial\Omega_I)$ (and $K_1^{S^1}(\partial\Omega_I)$). The quotient space $X_I := Z_I/S^1$ is an embedded smooth submanifold of \mathbb{CP}^{m-1} . Let D_{X_I} be the fundamental class in $K_0(X_I)$ associated to the $\bar{\partial}$ -operator on X_I . In Proposition 3.4, we explain that there is a natural isomorphism α_I from $K_1^{S^1}(\partial\Omega_I)$ to $K_0(X_I)$, mapping $\partial[D_N]$ to $[D_{X_I}]$. Let $\iota : \partial\Omega_I \hookrightarrow \partial\mathbb{B}^m$ (and $i : X_I \hookrightarrow \mathbb{CP}^{m-1}$) denote the embedding map. A good understanding of $\iota_*(\partial[D_N]) \in K_1^{S^1}(\mathbb{S}^{2m-1})$ will determine $i_*([D_{X_I}]) \in K_0(\mathbb{CP}^{m-1})$ completely. This provides an analytic approach to the Grothendieck-Riemann-Roch theorem for the projective variety X_I . In Theorem 1.2, we do not assume that the ideal I is homogeneous, and would like to view it as an analytic Grothendieck-Riemann-Roch Theorem.

Let ρ be a defining function on Ω_I , i.e. $\rho < 0$ on Ω_I , and $d\rho \neq 0$ on $\partial\Omega_I$. For $s \geq -1$, let $L_s^2(\Omega_I)$ be the (s) -weighted L^2 -space on Ω_I with the norm defined by

$$\|f\|_s^2 = \int_{\Omega_I - \Sigma_I} |f|^2 (-\rho)^s dV_I.$$

Let $L_{a,s}^2(\Omega_I)$ (and $L_{a,s}^p(\Omega_I)$) be the weighted Bergman space on Ω_I consisting of functions in $L_s^2(\Omega_I)$ (and $L_s^p(\Omega_I)$) that are holomorphic on Ω_I^0 . $L_{a,s}^2(\Omega_I)$ is naturally a Hilbert A -module. Recall that a Hilbert A -module (H, α) (i.e. $\alpha : A \rightarrow \mathcal{L}(H)$) is essentially normal

if $[\alpha(z_i), \alpha(z_j)^*]$ is compact for all $1 \leq i, j \leq m$. We have the following corollary from Theorem 1.2.

Corollary 1.3. *Under Assumption 1.1, for $s \geq 0$,*

- (1) *The A -module $L_{a,s}^2(\Omega_I)$ is essentially normal.*
- (2) *The A -module $L_{a,s}^2(\Omega_I)$ defines a K -homology class of the smooth boundary $\partial\Omega_I$, which is the fundamental class of $\partial\Omega_I$ defined by the CR -structure on $\partial\Omega_I$.*

As an application, we use our index theorem, in particular Corollary 1.3, to study Hilbert modules associated to I . Let $L_a^2(\mathbb{B}^m)$ be the Bergman space on \mathbb{B}^m . The Toeplitz operators make $L_a^2(\mathbb{B}^m)$ into an essentially normal Hilbert A -module. Observe that the restriction map R maps $f \in L_a^2(\mathbb{B}^m)$ to a holomorphic function $f|_{\Omega_I}$ on Ω_I . Let M be the complex codimension of Ω_I in \mathbb{B}^m . When Ω_I is smooth and intersects with $\partial\mathbb{B}^m$ transversely, Beatrous [7] proved that R maps $L_a^2(\mathbb{B}^m)$ continuously onto $L_{a,M}^2(\Omega_I)$. Using the developments in complex analysis [28], [35], we have the following generalization of Beatrous' result.

Theorem 1.4. *(Theorem 4.3) Under Assumption 1.1, there is a continuous linear operator $E : L_{a,M}^2(\Omega_I) \rightarrow L_a^2(\mathbb{B}^m)$ such that $RE = \text{Id}$. Therefore, the restriction operator R maps $L_a^2(\mathbb{B}^m)$ continuously onto $L_{a,M}^2(\Omega_I)$.*

By Assumption 1.1 and the dimension assumption $k \geq 2$, the Hartogs principle [27, Ch.III, Ex. 3.5] holds on Ω_I and states that every holomorphic function on Ω_I^0 is holomorphic on Ω_I . Furthermore, Assumption 1.1 plays a key role in the integral formula obtained in [28]. Hence Assumption 1.1 is crucial in Theorem 1.4 for R to be surjective. In general, without Assumption 1.1 the restriction map R may fail to be surjective (c.f. Sec. 5.1). We plan to study some cases in the near future when the range of R has finite codimension.

In Theorem 6.3, with Kai Wang we prove that, under Assumption 1.1, the kernel of the map R is the closure \bar{I} of I in $L_a^2(\mathbb{B}^m)$. Hence we have an exact sequence of Hilbert A -modules

$$(1) \quad 0 \longrightarrow \bar{I} \longrightarrow L_a^2(\mathbb{B}^m) \longrightarrow L_{a,M}^2(\Omega_I) \longrightarrow 0.$$

As the last two A -modules in the exact sequence (1) are essentially normal, we obtain the following theorem. We refer the reader to [18], and [24]–[26] for related results.

Theorem 1.5. *(Theorem 4.5) Under Assumption 1.1, both \bar{I} and the quotient $Q_I := L_a^2(\mathbb{B}^m)/\bar{I}$ are essentially normal A -modules. Furthermore, Q_I and $L_{a,M}^2(\Omega_I)$ correspond to the same class in $K_1(\partial\Omega_I)$.*

Theorem 1.5 confirms the conjecture by Arveson [2] and the first author [16] that the ideal \bar{I} is an essentially normal A -module when I satisfies Assumption 1.1. We refer

to Remark 4.6 for the discussion on the p -summability of the modules. This also suggests a good candidate for a fundamental class in $K_1(\partial\Omega_I)$ (and $K_0(X_I)$) when $\partial\Omega_I$ (and $X_I = \partial\Omega_I/S^1$) is not smooth. In algebraic geometry, when the zero variety Ω_I (and X_I) is not smooth, the Grothendieck-Riemann-Roch theorem usually involves resolutions of singularities by Hironaka's famous theorem [29]. Notice that resolutions [29] of a singularity variety are not unique. Therefore, it is hard to talk about a fundamental class on X_I from the algebraic geometric point of view. We observe that the quotient A -module Q_I is always well defined without any requirements on Ω_I . Theorem 1.5 suggests that $Q_I = L_a^2(\mathbb{B}^m)/\bar{I}$ may in general be a fundamental class in $K_1(\partial\Omega_I)$.

Remark 1.6. (1) *Assumption 1.1 can be weakened. For example, a natural case is that I is generated by a finite number of holomorphic functions that are defined in a neighborhood of the closed ball $\bar{\mathbb{B}}^m$ satisfying Assumption 1.1. All results in this article naturally extend to this case. We suggest the reader to compare our results to [18], where p is required to be a polynomial.*

(2) *Under Assumption 1.1, the ideal I is radical. The extensions of our results to non radical ideals will be reported in the near future.*

This paper is organized as follows. In Sec. 2, we will review some background knowledge about resolution of singularities, which is crucial in our proofs. In particular, we will explain the construction of the Hilbert space $L_{a,s}^2(\Omega_I)$. In Sec. 3, we will present the construction of the K -homology class D_N , and prove Theorem 1.2. In Sec. 4, we will prove Theorem 1.4 about the restriction map R and Theorem 1.5 about the A -module structures on the ideal \bar{I} and the quotient Q_I . We end this article with two remarks in Sec. 5. In the Appendix we prove together with Kai Wang that under Assumption 1.1, the closure \bar{I} of I in $L_a^2(\mathbb{B}^m)$ agrees with the kernel of the operator R .

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2. INTEGRATION ON THE ZERO VARIETY

In this section, we provide a preliminary review of resolution of singularities and the construction of the Hilbert space $L_{a,s}^2(\Omega_I)$.

2.1. Resolution of singularities. We recall some useful properties about resolution of Ω_I . Hironaka [29] proved that every algebraic variety V over \mathbb{C} has a resolution. We will use the resolution method to study the zero variety Ω_I . More explicitly, there is a smooth manifold $\tilde{\Omega}_I$ with a proper holomorphic surjection $\pi : \tilde{\Omega}_I \rightarrow \Omega_I$ with the following properties:

- (1) The exceptional set $E_I := \pi^{-1}(\Sigma_I)$ is a hypersurface in Ω_I with (possible) “normal crossing singularities” only.
- (2) The restriction of $\pi : \tilde{\Omega}_I - E_I \rightarrow \Omega_I - \Sigma_I$ is a biholomorphism.

The pullback $\pi^*\sigma_I$ is a positive semidefinite metric on $\tilde{\Omega}_I$ degenerated on E_I . The pullback π^*dV_I is a volume element on $\tilde{\Omega}_I$ that vanishes on E_I . We choose a hermitian metric σ on $\tilde{\Omega}_I$. And denote dV_σ to be the associated volume element on $\tilde{\Omega}$. Define $d_{E_I}(x)$ to be the distance function on $\tilde{\Omega}_I$ from x to the exceptional subset E_I . In [22, Eq. (9)], it is proved that there are positive constant c, C, M such that

$$(2) \quad cd_{E_I}(x)^M dV_\sigma \leq \pi^*dV_I \leq CdV_\sigma, \quad \text{on } \tilde{\Omega}_I - E_I.$$

Without loss of generality, we may assume that E_I is a divisor with only normal crossing, i.e. the irreducible components of E are regular and meet complex transversely. As is explained in [32, Sec. 3], by Eq. (2) there is an effective divisor D of $\tilde{\Omega}_I$ that is supported on E_I such that for (p, q) -forms $\Omega^{p,q}$ on U ,

$$(3) \quad L^2(U, dV_\sigma, \Omega^{p,q} \otimes L_{-D}) \subset L^2(U, \pi^*dV_I, \Omega^{p,q}) \subset L^2(U, dV_\sigma, \Omega^{p,q} \otimes L_D),$$

for U a neighborhood of E_I in $\tilde{\Omega}_I$.

2.2. Weighted Bergman space. Let $L_{a,s}^2(\Omega_I)$ be the space of holomorphic functions on $\Omega_I - \Sigma$ that are square integrable with respect to the measure $(-\rho)^s dV_I$.

Lemma 2.1. $L_{a,s}^2(\Omega_I)$ is a closed subspace of $L_s^2(\Omega_I)$, and therefore a Hilbert space.

Proof. Let $L_s^2(\Omega_I)$ be the s -weighted L^2 -space on Ω_I with respect to $(-\rho)^s dV_I$. Pull back the space $L_s^2(\Omega_I)$ to the resolution $\tilde{\Omega}_I$. The pullback map is an isomorphism from $L_s^2(\Omega_I)$ to $L_s^2(\tilde{\Omega}_I, \pi^*dV_I)$. By the inequality (2) and (3), we have the following inclusion

$$(4) \quad L_s^2(\tilde{\Omega}_I, dV_\sigma, L_{-D}) \subset L_s^2(\tilde{\Omega}_I, \pi^*dV_I) \subset L_s^2(\tilde{\Omega}_I, dV_\sigma, L_D).$$

Consider the $\bar{\partial}$ -equation $\bar{\partial}\varphi = 0$ on $\tilde{\Omega}_I$. Let $\ker \bar{\partial}|_{L_{-D}}$ (and $\ker \bar{\partial}|_{L_D}$) be the solution space of the $\bar{\partial}$ -operator in $L_s^2(\tilde{\Omega}_I, dV_\sigma, L_{-D})$ (and $L_s^2(\tilde{\Omega}_I, dV_\sigma, L_D)$). Let $\ker \bar{\partial}|_{\Omega_I}$ be the

solution space of the $\bar{\partial}|_{\Omega_I}$ operator in $L_s^2(\Omega_I)$ and therefore $L_s^2(\tilde{\Omega}_I, \pi^*dV_I)$. The following inclusion follows directly from (4),

$$\ker \bar{\partial}|_{L_{-D}} \subset \ker \bar{\partial}|_{\Omega_I} \subset \ker \bar{\partial}|_{L_D}.$$

Next consider the inclusion map

$$i_* : \ker \bar{\partial}|_{L_{-D}} \subset \ker \bar{\partial}|_{L_D}.$$

Following the proof of [32, Theorem 3.1], on a pseudoconvex neighborhood W of a connected component of E_I , one has the following exact sequence

$$0 \longrightarrow \Gamma(W, L_{-D}) \xrightarrow{i_*|_W} \Gamma(W, L_D) \longrightarrow \Gamma(W, Q),$$

where $Q := L_D/(i_*L_{-D})$ is a coherent analytic sheaf with compact support in W . As Q is compactly supported, $\Gamma(W, Q)$ is of finite dimension. Therefore, $i_*|_W(\Gamma(W, L_{-D}))$ is of finite codimension in $\Gamma(W, L_D)$. Globally, as E_I only has finitely many components, $i_* : \ker \bar{\partial}|_{L_{-D}} \subset \ker \bar{\partial}|_{L_D}$ is of finite codimension.

As $\ker \bar{\partial}|_{\Omega_I}$ is between $\ker \bar{\partial}|_{L_{-D}}$ and $\ker \bar{\partial}|_{L_D}$, $L_{a,s}^2(\Omega_I) \cong L_{a,s}^2(\tilde{\Omega}_I, \pi^*dV_I)$ is of finite codimension in $L_{a,s}^2(\tilde{\Omega}_I, dV_\sigma, L_D)$. Therefore, $L_{a,s}^2(\Omega_I)$ is a closed subspace of $L_{a,s}^2(\tilde{\Omega}_I, dV_\sigma, L_D)$ of finite codimension. As $L_{a,s}^2(\tilde{\Omega}_I, dV_\sigma, L_D)$ is closed in $L_s^2(\tilde{\Omega}_I, dV_\sigma, L_D)$, we conclude that $L_{a,s}^2(\Omega_I)$ is a closed subspace of $L_s^2(\Omega_I) = L_s^2(\tilde{\Omega}, \pi^*dV_I)$ from Eq. (4), and therefore a Hilbert space. \square

Remark 2.2. *Similar arguments to the proof of Lemma 2.1 confirm that $L_{a,s}^p(\Omega_I)$ is a Banach space for $p \geq 1, s \geq -1$.*

3. AN ODD INDEX THEOREM FOR ANALYTIC SPACE WITH ISOLATED SINGULARITY

In this section, we explain the construction of the operator D_N on Ω_I^0 with the Neumann boundary condition and present the proof of Theorem 1.2.

In [9], Boutet de Monvel proved an index theorem for Toeplitz operators on a complex manifold with strongly pseudoconvex boundary. Our result can be viewed as an extension of Boutet de Monvel's theorem to complex manifolds with isolated singularities. Such an extension was hinted by Boutet de Monvel [9] and an approach was explained to the second author [10]. In the following development, we will take a different route by following the relative K -homology theory developed by Baum-Douglas-Taylor [4]. For simplicity, we will present our proofs below with the standard volume dV_I on Ω_I . The same results also hold true for the weighted volume element $(-\rho)^s dV_I$ ($s \geq 0$) with similar arguments. We also point out that although we have used the notation Ω_I , the results in this section hold true for more general complex analytic spaces (See Remark 3.3).

3.1. The $\bar{\partial}$ -equation on Ω_I^0 . Let Ω_I^0 denote $\Omega_I - \Sigma_I$. Let $\wedge^{p,q} T^* \Omega_I^0$ be the degree (p, q) subbundle of $\wedge^{p+q} T^* \Omega_I^0$. Recall that σ_I is the subspace metric on $\Omega_I^0 \subset \mathbb{B}^m$. Denote $L^{p,q}(\Omega_I^0)$ to be the L^2 -space on Ω_I^0 associated to the metric σ_I . Consider the $\bar{\partial}_{p,q}$ -operator on $L^{p,q}(\Omega_I^0)$

$$\bar{\partial}_{p,q} : L^{p,q}(\Omega_I^0) \rightarrow L^{p,q+1}(\Omega_I^0),$$

and its adjoint operator

$$\bar{\partial}_{p,q}^* : L^{p,q+1}(\Omega_I^0) \rightarrow L^{p,q}(\Omega_I^0).$$

Let r be a real valued function smooth in a neighborhood of $\partial\Omega_I$ satisfying $r = 0$ on $\partial\Omega_I$ and $dr \neq 0$ on $\partial\Omega_I$. Define the $\bar{\partial}$ -Neumann boundary condition by

$$\mathcal{D}^{p,q} := \{\xi \in \Gamma(\wedge^{p,q} T^*(\bar{\Omega}_I - \Sigma_I)) : (\bar{\partial}r) \lrcorner \xi = 0 \text{ on } \partial\Omega_I\}.$$

When $q = 0$, the $\bar{\partial}$ -Neumann boundary condition is trivial. Let $\bar{\partial}_{p,q}^N$ denote the closure of $\bar{\partial}_{p,q}$ restricted to $\mathcal{D}^{p,q}$. Consider the Laplace operator by

$$\square_{p,q}^N = \bar{\partial}_{p,q-1}^N (\bar{\partial}_{p,q-1}^N)^* + (\bar{\partial}_{p,q}^N)^* \bar{\partial}_{p,q}^N : L^{p,q}(\Omega_I^0) \longrightarrow L^{p,q}(\Omega_I^0).$$

Since $\partial\Omega_I$ is strongly pseudoconvex, the Laplace operator $\square_{0,q}^N$ ($q \geq 1$) with the $\bar{\partial}$ -Neumann boundary condition has compact resolvent on the resolution $\tilde{\Omega}_I$. Hence, [32, Theorem 1.1] implies that for $q \geq 1$,

$$\square_{0,q}^N : L^{0,q}(\Omega_I^0) \rightarrow L^{0,q}(\Omega_I^0)$$

has compact resolvent.

We remark that when the dimension of Ω_I is greater than or equal to 2, under Assumption 1.1 the Hartogs principle [27, Ch.III, Ex.3.5] implies that every function in $\ker \bar{\partial}_{0,0}^N = L_a^2(\Omega_I)$ is holomorphic on the whole Ω_I .

3.2. A Hilbert module. In this section, following [4], we construct a K -homology class D_N in $K_0(\bar{\Omega}_I, \partial\Omega_I)$.

Let $H_0(\Omega_I^0) = \oplus_{q \geq 0} L^{0,2q}(\Omega_I^0)$ (and $H_1(\Omega_I^0) = \oplus_{q \geq 0} L^{0,2q+1}(\Omega_I^0)$) be the Hilbert space of $(0, \text{even})$ (and $(0, \text{odd})$) forms on Ω_I^0 . We consider the differential operator

$$D_N := \bar{\partial}^N + (\bar{\partial}^N)^* : H_0(\Omega_I^0) \longrightarrow H_1(\Omega_I^0).$$

The operator D_N is a first order differential operator on Ω_I . Denote $\sigma_{D_N}(x, \xi)$ to be the symbol of D_N . Let $\mathcal{D}(D_N)$ denote the domain of D_N in $H_0(\Omega_I^0)$.

Let $C(\bar{\Omega}_I)$ be the C^* -algebra of continuous functions on the closure $\bar{\Omega}_I$. Consider the multiplication of $C(\bar{\Omega}_I)$ on H_0 and H_1 . Denote the corresponding $*$ -representations by $\sigma_0 : C(\bar{\Omega}_I) \rightarrow \mathcal{L}(H_0)$ and $\sigma_1 : C(\bar{\Omega}_I) \rightarrow \mathcal{L}(H_1)$.

Proposition 3.1. *The graded Hilbert space $H := H_0 \oplus H_1$, and the representation*

$$\sigma := \sigma_0 \oplus \sigma_1 : C(\overline{\Omega}_I) \longrightarrow \mathcal{L}(H),$$

and the operator

$$A = \begin{pmatrix} 0 & D_N^* \\ D_N & 0 \end{pmatrix}$$

form an unbounded Kasparov module for $C(\overline{\Omega}_I)$ and its ideal $C_0(\overline{\Omega}_I)$ which consists of functions in $C(\overline{\Omega}_I)$ vanishing at the boundary $\partial\Omega_I$.

Proof. Let $C^\infty(\overline{\Omega}_I)$ be the space of continuous functions on $\overline{\Omega}_I$ whose pullback to the resolution $\tilde{\Omega}_I$ via the map π is smooth on the closure $\overline{\tilde{\Omega}_I}$. Since the singularities on Ω_I are isolated, by a partition of unity, we can easily show that $C^\infty(\overline{\Omega}_I)$ is a dense $*$ -subalgebra of $C(\overline{\Omega}_I)$. One quickly checks that for any $u \in H_{loc}^1(\Omega_I^0, \wedge^{0,even} T^* \Omega_I^0)$

$$D_N \sigma_0(f)u - \sigma_1(f)D_N u = \sigma_{D_N}(x, df)u, \quad \forall f \in C^\infty(\overline{\Omega}_I).$$

As the metric on Ω_I^0 vanishes toward singular points, $\sigma_{D_N}(x, df)$ extends to a bounded linear operator from H_0 to H_1 . From [4, Prop. 1.3], we can conclude that for every $f \in C^\infty(\overline{\Omega}_I)$, $\sigma_1(f)$ preserves the domain of $D_N^* : H_1(\Omega_I^0) \rightarrow H_0(\Omega_I^0)$, and $[\sigma(f), D_N]$ extends to a bounded operator on $H_0(\Omega_I^0) \oplus H_1(\Omega_I^0)$. And, therefore $[\sigma(f), D_N]$ is bounded for all $f \in C(\overline{\Omega}_I)$.

As the resolution $\tilde{\Omega}_I$ is a complex manifold with strongly pseudoconvex boundary, Kohn [31] showed that the $\bar{\partial}$ -laplacian $\square_{0,q}^N$ has a finite dimensional kernel and a compact solution operator on $L^{0,q}(\tilde{\Omega}_I)$ for all $q \geq 1$. By [32, Thm. 1.2], the $\bar{\partial}$ -laplacian $\square_{0,q}^N$ also has a finite dimensional kernel and a compact solution operator on $L^{0,q}(\Omega_I^0)$, for $q \geq 1$. From this, we can derive that $\square_{0,q}^N$ on $L^{0,q}(\Omega_I^0)$ has a compact resolvent for $q \geq 1$. Therefore, the operator

$$D_N D_N^* : H_1(\Omega_I^0) \rightarrow H_0(\Omega_I^0)$$

has a compact resolvent.

For $f \in C_0^\infty(\overline{\Omega}_I)$, the operator $\sigma_0(f)(D_N^* D_N + 1)^{-1}$ maps $L^2(\Omega_I^0, \wedge^{0,even} T^* \Omega_I^0)$ to the Sobolev space $H_0^{2,2}(\Omega_I^0, \wedge^0 T^* \Omega_I^0)$, where $H_0^{2,2}$ is the L^2 -Sobolev space of sections that vanishes on $\partial\Omega_I$. Hence the operator $\sigma_0(f)(D_N^* D_N + 1)^{-1}$ is compact on $L^2(\Omega_I^0, \wedge^{0,even} T^* \Omega_I^0)$ by the Rellich¹ compact embedding theorem. Analogously, the operator $\sigma_1(f)(D_N D_N^* + 1)^{-1}$ is compact on $L^2(\Omega_I^0, \wedge^{0,odd} T^* \Omega_I^0)$.

By [4, Prop. 1.1, 1.39, Prop. 3.1], we conclude (H, σ, A) is an unbounded Kasparov module. \square

¹Recall that the volume element on Ω_I^0 vanishes at the singular points of a certain order, i.e. Eq. (2). Using this fact, one can prove a Rellich compact embedding theorem for $H_0^{2,2}(-) \hookrightarrow L^2(-)$ in the same way as [21].

Besides the $\bar{\partial}$ -Neumann boundary condition, we can consider other boundary conditions. For example, denote D_{max} and D_{min} to be the maximal and minimal extension of the first order differential operator $D = \bar{\partial} + \bar{\partial}^*$ on $C_c^\infty(\Omega_I^0, \wedge^{0,even} T^* \Omega_I^0)$. More explicitly, the domain $\mathcal{D}(D_{max})$ is

$$\{\xi \in L^2(\Omega_I^0, \wedge^{0,even} T^* \Omega_I^0) : D\xi \in L^2(\Omega_I^0, \wedge^{0,odd} T^* \Omega_I^0)\}.$$

Let $D^t : C_c^\infty(\Omega_I^0) \rightarrow C_c^\infty(\Omega_I^0)$ be the formal adjoint of D . We have $D_{max}^t = (D_{min})^*$, and

$$D_{min} \subset D_N \subset D_{max}.$$

Proposition 3.2. *The operator D_{max} (and D_{min}) on (H, σ) defines a K -cycle $[D_{max}]$ (and $[D_{min}]$) for $K_0(\overline{\Omega}_I, \partial\Omega_I)$. Furthermore, in $K_0(\overline{\Omega}_I, \partial\Omega_I)$*

$$[D_{max}] = [D_{min}] = [D_N].$$

The proof of Proposition 3.2 is a straightforward extension of [4, Prop. 2.1, 3.1, 3.3]. We skip the detail to avoid repetition.

3.3. The boundary map in K-homology. In [3], Baum and the first author developed a long exact sequence for relative K -homology. In particular, applying the long exact sequence to our study, we obtain a boundary map $\partial : K_0(\overline{\Omega}_I, \partial\Omega_I) \rightarrow K_1(\partial\Omega_I)$. In this subsection, we study the boundary $\partial[D_N] \in K_1(\partial\Omega_I)$.

Let $\ker(D_N)$ be the space

$$\{\xi \in L^2(\Omega_I^0, \wedge^{0,even} T^* \Omega_I^0) : D_N(\xi) = 0\}.$$

By the property that D_N has a finite dimensional solution space on $L^2(\Omega_I^0, \wedge^{0,q} T^* \Omega_I^0)$ for $q \geq 1$, we know that up to a finite dimensional subspace $\ker(D_N)$ is equal to $L_a^2(\Omega_I)$, the space of L^2 -holomorphic functions on Ω_I^0 . The K -homology class in $K_1(\partial\Omega_I)$ associated to $\ker(D_N)$ is equal to the K -homology class associated to $L_a^2(\Omega_I)$, i.e.

$$[\ker(D_N)] = [L_a^2(\Omega_I)].$$

As $\partial\Omega_I$ is a strongly pseudoconvex, the restriction of the complex structure on Ω_I to the boundary defines a CR -structure on $\partial\Omega_I$, and therefore a spin^c structure on $\partial\Omega_I$. Let $D_{\partial\Omega_I}$ be the Dirac operator associated to this CR -structure. Then we can conclude from [4, Prop. 4.5, 4.6] that in $K_1(\partial\Omega_I)$

$$(5) \quad \partial[D_N] = [\ker D_N] = [L_a^2(\Omega_I)] = [D_{\partial\Omega_I}].$$

3.4. Proof Theorem 1.2 and Corollary 1.3. Theorem 1.2 is a corollary of Proposition 3.1 and Eq. (5). We explain the proof of Corollary 1.3. By Proposition 3.1, 3.2, and Eq. (5), we conclude that the $L_a^2(\Omega_I)$ defines a K -homology class on $\partial\Omega_I$. This implies that $L_a^2(\Omega_I)$ is an essentially normal A -module, and confirms Part (i) of Theorem 1.2. Also we conclude from Eq. (5) that $[L_a^2(\Omega_I)]$ is equal to the fundamental class of $\partial\Omega_I$ associated to the canonical CR -structure and therefore the contact structure on $\partial\Omega_I$, and confirms Part (ii) of Theorem 1.2.

Remark 3.3. *We point out that the proofs of Theorem 1.2 only use the property that Ω_I is a complex analytic space of pure dimension n with the following properties.*

- (1) Ω_I has a strongly pseudoconvex boundary $\partial\Omega_I := \overline{\Omega_I} \setminus \Omega_I$.
- (2) Ω_I may contain isolated singularities away from $\partial\Omega_I$.

From these assumptions, we can conclude from [32, Theorem 1.2] that that operator $D_N D_N^$ has compact resolvents and therefore Proposition 3.1, 3.2, and Theorem 1.2.*

When I is a homogeneous ideal of $A = \mathbb{C}[z_1, \dots, z_m]$, the zero variety Z_I is invariant under the multiplication by $\mathbb{C}^* = \mathbb{C} - \{0\}$. Assume that the origin is the only possible isolated singular point of Z_I . Z_I intersects with the unit sphere \mathbb{S}^{2m-1} transversely, and therefore the boundary $\partial\Omega_I$ is a smooth submanifold of \mathbb{S}^{2m-1} . The group S^1 as the unit circle in \mathbb{C}^* acts on the unit sphere \mathbb{S}^{2m-1} freely. As I is a homogeneous ideal, $\partial\Omega_I$ is invariant under the S^1 -action. We easily observe that the K -homology class $[D_N]$ (and $\partial[D_N]$) is S^1 -equivariant and lives in $K_0^{S^1}(\overline{\Omega_I}, \partial\Omega_I)$ (and in $K_1^{S^1}(\partial\Omega_I)$). Furthermore, the quotient space $X_I := \partial\Omega_I/S^1$ is an embedded smooth submanifold of \mathbb{CP}^{m-1} , which is the projective variety associated to the ideal I . Let $D_{X_I} \in K_0(X_I)$ be the fundamental class on X_I associated to the $\bar{\partial}$ -operator. We explain below the relation between $\partial[D_N] \in K_1^{S^1}(\partial\Omega_I)$ and $[D_{X_I}] \in K_1(X_I)$.

Proposition 3.4. *When the ideal I is homogeneous and the origin is the only possible singular point of Z_I , there is a natural isomorphism α_I from $K_1^{S^1}(\partial\Omega_I)$ to $K_0(X_I)$ such that $\alpha_I(\partial[D_N]) = [D_{X_I}]$ in $K_0(X_I)$.*

Proof. We observe that as I is homogeneous, the CR -structure on $\partial\Omega_I$ is S^1 -equivariant, and gives an S^1 -equivariant spin^c structure on $\partial\Omega_I$. As $\dim(\partial\Omega_I)$ is odd, the S^1 -equivariant Poincaré duality gives an isomorphism

$$(PD_{\partial\Omega_I}^{S^1})^{-1} : K_1^{S^1}(\partial\Omega_I) \xrightarrow{\cong} K_0^{S^1}(\partial\Omega_I).$$

As the S^1 -action on $\partial\Omega_I$ is free, $K_{S^1}^0(\partial\Omega_I)$ is naturally isomorphic to $K^0(X_I)$. Let β_I denote the isomorphism from $K_{S^1}^0(\partial\Omega_I)$ to $K^0(X_I)$. X_I is a complex manifold with a canonical spin^c structure. The Poincaré duality gives an isomorphism

$$PD_{X_I} : K^0(X_I) \longrightarrow K_0(X_I).$$

Define $\alpha_I : K_1^{S^1}(\partial\Omega_I) \rightarrow K_0(X_I)$ to be the composition $PD_{X_I} \circ \beta_I \circ (PD_{\partial\Omega_I}^{S^1})^{-1}$. α_I is obviously an isomorphism as each involved component is.

Theorem 1.2 identifies the class $\partial[D_N]$ with the fundamental class $D_{\partial\Omega_I}$ associated to the CR -structure on $\partial\Omega_I$. Furthermore, it is not hard to trace through the arguments that this is an identification in $K_1^{S^1}(\partial\Omega_I)$. The inverse Poincaré duality map $(PD_{\partial\Omega_I}^{S^1})^{-1}$ maps $D_{\partial\Omega_I}$ to the trivial line bundle $\mathbb{C}_{\partial\Omega_I}$ in $K_{S^1}^0(\partial\Omega_I)$. The map β_I maps $\mathbb{C}_{\partial\Omega_I}$ to the trivial line bundle \mathbb{C}_{X_I} in $K^0(X_I)$. And the Poincaré duality map PD_{X_I} on X_I maps \mathbb{C}_{X_I} to the fundamental class D_{X_I} in $K_0(X_I)$. Therefore, we conclude that α_I maps $\partial[D_N]$ to $[D_{X_I}]$ in $K_0(X_I)$. \square

4. GEOMETRIZATION OF THE QUOTIENT HILBERT MODULE

In this section, we construct a right inverse E of the restriction operator $R : L_a^2(\mathbb{B}^m) \rightarrow L_{a,M}^2(\Omega_I)$, and apply it to prove that both the closure \bar{I} of I in $L_a^2(\mathbb{B}^m)$ and the quotient $Q_I = L_a^2(\mathbb{B}^m)/\bar{I}$ are essentially normal A -modules. We prove a Toeplitz index theorem for Q_I by identifying the K -homology class associated to Q_I with the fundamental class on $\partial\Omega_I := \bar{\Omega}_I \setminus \Omega_I$.

4.1. Integral representation formula. In [28, Theorem I1], an integral representation formula was obtained for an analytic space satisfying Assumption 1.1. More precisely, let α be the volume form on $\partial\Omega_I$. There is a kernel function $K(z, \zeta)$ on $\Omega_I \times \partial\Omega_I$ such that for each $\zeta \in \partial\Omega_I$, the function $K(-, \zeta)$ is holomorphic on $\bar{\Omega}_I$. Let f be a holomorphic function on $\bar{\Omega}_I$. Then f can be represented by the following integral

$$(6) \quad f(z) = \int_{\partial\Omega_I} f(\zeta) \frac{K(z, \zeta)}{(1 - \sum_{i=1}^m \bar{\zeta}_i z_i)^k} \alpha(\zeta).$$

By choosing a cut-off function χ supported in a neighborhood of the boundary $\partial\Omega_I$, we can conclude the following corollary from Equation (6).

Lemma 4.1. *Under Assumption 1.1, there is neighborhood M_I of Ω_I in Z_I , and a smooth differential form $\eta_j(z, \zeta)$ ($j = 0, 1, \dots$) on $M_I \times M_I$ of bidegree (k, k) (when $j = 0$, $(k, k-1)$) in ζ supported away from Σ_I , the set of singular points, and $(0, 0)$ in z such that $\eta_j(-, \zeta)$ is holomorphic on M_I for any $\zeta \in M_I$. For any $f \in L_{a,s-1}^1(\Omega_I)$, the following integral representations hold,*

$$f(z) = \int_{\partial\Omega_I} \frac{f(\zeta) \eta_0(z, \zeta)}{(1 - \sum_{i=1}^m z_i \bar{\zeta}_i)^k}, \quad s = 0,$$

and

$$f(z) = \int_{\Omega_I} \frac{f(\zeta) \eta_s(z, \zeta) (-\rho)^{s-1}(\zeta)}{(1 - \sum_{i=1}^m z_i \bar{\zeta}_i)^{k+s}}, \quad s \geq 1.$$

Proof. The proof is a line by line repetition of the proof of [7, Corollary 2.4] from the representation (6). It is worth pointing out that the cut-off function χ in the proof of [7, Corollary 2.4] can be chosen to be supported away from the set Σ_I of singular points in Ω_I . Therefore, the support of $\eta_j(-, \zeta)$ is also away from Σ_I . We remark that the property that the dimension k of Ω_I is at least 2 and Assumption 1.1 assures that the Hartogs principle [27, Ch.III, Ex.3.5] holds on Ω_I . The Hartogs principle assures that any $f \in L^1_{a,s}(\Omega_I)$ is holomorphic on Ω_I . Therefore, the singularities in Ω_I do not affect any parts of the proofs in [7]. We leave the detail to the reader. \square

4.2. Extension operator. Let $C(\mathbb{B}^m)$ (and $C(\Omega_I)$) be the space of continuous functions on \mathbb{B}^m (and Ω_I). Consider the restriction operator $R : C(\mathbb{B}^m) \rightarrow C(\Omega_I)$ defined by

$$R(f)(z) := f(z), \text{ for } z \in \Omega_I.$$

The restriction of R to the subspace $\mathcal{O}(\mathbb{B}^m)$ of holomorphic functions on \mathbb{B}^m takes image in $\mathcal{O}(\Omega_I)$.

The function $\eta_0(-, \zeta)$ can be viewed as a holomorphic function from Ω_I to smooth $(k, k-1)$ forms on M_I . By [13, Corollary 12.1], $\eta_0(-, \zeta)$ can be extended to a holomorphic function on \mathbb{B}^m with value in smooth $(k, k-1)$ forms on M_I supported away from the set Σ_I of singular points, which will be again denoted by $\eta_0(z, \zeta)$. Define a linear operator $E_0 : L^1_{a,-1}(\Omega_I) \rightarrow \mathcal{O}(\mathbb{B}^m)$ by

$$E_0(f)(z) = \int_{\partial\Omega_I} \frac{f^*(\zeta)\eta_0(z, \zeta)}{(1 - \sum_{i=1}^m z_i \bar{\zeta}_i)^k},$$

where f^* is the boundary value of f .

Similarly we extend $\eta_j(z, \zeta)$ to a holomorphic function on \mathbb{B}^m with value in smooth (k, k) forms on M_I (a neighborhood of Ω_I in Z_I) supported away from Σ_I . Define a linear operator $E_j : L^1_{a,j}(\Omega_I) \rightarrow \mathcal{O}(\mathbb{B}^m)$ by

$$E_j f(z) = \int_{\Omega_I} \frac{f(\zeta)\eta_j(z, \zeta)(-\rho)^{j-1}(\zeta)}{(1 - \sum_{i=1}^m z_i \bar{\zeta}_i)^{k+j}}, \quad j \geq 1.$$

A direction generalization of [7, Theorem 2.5, Corollary 2.6] proves the following proposition.

Proposition 4.2. (1) For every $f \in L^1_{a,-1}(\Omega_I)$, $E_0(f) \in \mathcal{O}(\overline{\mathbb{B}^m} \setminus \partial\Omega_I)$ and $E_0(f)|_{\Omega_I} = f$.

(2) For $j \geq 1$, every $f \in L^1_{a,j-1}(\Omega_I)$, $E_j(f) \in \mathcal{O}(\mathbb{B}^m)$ and

$$E_j f|_{\Omega_I} = f, \quad E_j f = E_k f, \quad k < j.$$

The following theorem is a strengthened version of Theorem 1.4.

Theorem 4.3. *Under Assumption 1.1, there is a continuous linear operator*

$$E : L_{a,s+M}^p(\Omega_I) \longrightarrow L_{a,s}^p(\mathbb{B}^m), \quad 1 < p < \infty, -1 \leq s,$$

such that $RE = Id$. Therefore, the restriction operator R is a surjective bounded linear operator

$$R : L_{a,s}^p(\mathbb{B}^m) \longrightarrow L_{a,s+M}^p(\Omega_I), \quad 1 < p < \infty, -1 \leq s.$$

Proof. The proof is a direct generalization of the proof [7, Thm. 1.1]. The key observation is that with the choice of the cut-off function χ in the construction of Lemma 4.1, $\eta_j(-, \zeta)$ ($j \geq 0$) is zero outside a neighborhood of the boundary $\partial\Omega_I$. Therefore, the support of $\eta_j(z, \zeta)$ is away from singular ζ values in Ω_I . Hence, the kernel function $\frac{\eta_j(z, \zeta)\rho^{j-1}(\zeta)}{(1 - \sum_{i=1}^m z_i \bar{\zeta}_i)^{k+j}}$ is bounded on the whole Ω_I by

$$\frac{CdV_I}{(1 - \sum_{i=1}^m z_i \bar{\eta}_i)^{k+j}}$$

for some constant $C > 0$. This property allows us to use [7, Theorem 4.1] to conclude the desired statements on the bounds of the operator R . \square

4.3. Equivalence of Hilbert modules. We look at the restriction operator

$$R : L_a^2(\mathbb{B}^m) \longrightarrow L_{a,M}^2(\Omega_I).$$

As R maps all functions in \bar{I} to the zero function on Ω_I , R restricts to a bounded linear map

$$R_{\Omega_I} : Q_I = L_a^2(\mathbb{B}^m)/\bar{I} \longrightarrow L_{a,M}^2(\Omega_I).$$

The extension map E with $RE = I$ implies that R_{Ω_I} is surjective.

To study the above structure, we prove the following general fact.

Proposition 4.4. *Let H_1 and H_2 be two Hilbert A -modules. Let $X : H_1 \rightarrow H_2$ be an isomorphism of A -modules. Any two of the following three statements imply the third one.*

- (1) *The A -module H_1 is essentially normal.*
- (2) *The A -module H_2 is essentially normal.*
- (3) *The operator $X^*X : H_1 \rightarrow H_1$ commutes with the A -module structure up to compact operators.*

Therefore, when H_1 and H_2 are isomorphic essentially normal A -modules, the corresponding extensions associated to H_1 and H_2 are unitarily equivalent.

Proof. By polar decomposition, we write

$$X = US,$$

where $S : H_1 \rightarrow H_1$ is an invertible positive operator, and $U : H_1 \rightarrow H_2$ is a unitary operator.

We observe that X satisfies that for any $p \in A$ and $\xi \in H_1$,

$$\sigma_2(p)X(\xi) = X\sigma_1(p)(\xi),$$

where $\sigma_i(p)$ is the representation of p on H_i . By the polar decomposition of X , we can write

$$(7) \quad \sigma_2(p) = X\sigma_1(p)X^{-1} = US\sigma_1(p)S^{-1}U^*.$$

We compute X^*X by

$$X^*X = SU^*US = S^2.$$

If $X^*X = S^2$ commutes with σ_1 up to compact operators, S commutes with σ_1 up to compact operators too. Equation (7) implies that in the Calkin algebra $\mathcal{C}(H_1)$,

$$(8) \quad \overline{\sigma_2(p)} = \overline{US\sigma_1(p)S^{-1}U^*} = (\overline{U})(\overline{\sigma_1(p)})(\overline{U})^*,$$

where we have used \overline{T} to denote the image of an operator T in the corresponding Calkin algebra.

The Equation (8) shows that the unitary operator U is a unitary equivalence between $\overline{\sigma_1}$ and $\overline{\sigma_2}$. From this we can conclude that

$$(1) \ \& \ (3) \implies (2), \text{ and } (2) \ \& \ (3) \implies (1).$$

In the following we show that

$$(1) \ \& \ (2) \implies (3).$$

By (1) and (2), we can extend $\overline{\sigma_1} : A \rightarrow \mathcal{C}(H_1)$ and $\overline{\sigma_2} : A \rightarrow \mathcal{C}(H_2)$ to $*$ -algebra morphisms

$$\overline{\sigma_1} : C(\overline{\mathbb{B}}^m) \rightarrow \mathcal{C}(H_1), \text{ and } \overline{\sigma_2} : C(\overline{\mathbb{B}}^m) \rightarrow \mathcal{C}(H_2).$$

By the Fuglede-Putnam theorem and the assumption that both $\sigma_1(p)$ and $\sigma_2(p)$ are essentially normal for $p \in A$, Equation (7) implies that

$$\overline{\sigma_2(p)}^* = \overline{X}\overline{\sigma_1(p)}^*\overline{X}^{-1},$$

and therefore

$$\overline{\sigma_2(p^*)} = \overline{X}\overline{\sigma_1(p^*)}\overline{X}^{-1}.$$

We conclude that for any $a \in C(\overline{\mathbb{B}}^m)$,

$$(9) \quad \overline{\sigma_2(a)} = \overline{X}\overline{\sigma_1(a)}\overline{X}^{-1} = (\overline{U})(\overline{S})\overline{\sigma_1(a)}(\overline{S})^{-1}(\overline{U})^*.$$

Taking the adjoint of the both sides of Eq. (9), we have

$$(10) \quad \overline{\sigma_2(a^*)} = \overline{\sigma_2(a)}^* = (\overline{X}^{-1})^*\overline{\sigma_1(a)}^*(\overline{X})^* = (\overline{U})(\overline{S})^{-1}\overline{\sigma_1(a^*)}(\overline{S})(\overline{U})^*, \text{ for all } a \in C(\overline{\mathbb{B}}^m).$$

Comparing Eq. (9) and Eq. (10), we obtain the following equality, for all $a \in C(\overline{\mathbb{B}}^m)$

$$(\overline{S})\overline{\sigma_1(a)}(\overline{S})^{-1} = (\overline{S})^{-1}\overline{\sigma_1(a)}(\overline{S}),$$

and

$$(\overline{S})^2 \overline{\sigma}_1(a) = \overline{\sigma}_1(a) (\overline{S})^2.$$

We conclude from the last equality that $\sigma_1(a)$ commutes with $S^2 = X^*X$ up to compact operators. And therefore $\overline{\sigma}_1$ and $\overline{\sigma}_2$ are unitarily equivalent by (8). \square

Notice that both Q_I and $L_{a,M}^2(\Omega_I)$ are A -modules. Denote σ_{Q_I} and σ_{Ω_I} to be the corresponding morphisms from $C(\overline{\mathbb{B}}^m)$ to $\mathcal{C}(Q_I)$ and $\mathcal{C}(\Omega_I)$, where $\mathcal{C}(Q_I)$ and $\mathcal{C}(\Omega_I)$ are the Calkin algebras on Q_I and $L_{a,M}^2(\Omega_I)$. Next theorem studies the relation between the two extension classes σ_{Q_I} and σ_{Ω_I} .

Theorem 4.5. *Under Assumption 1.1, both \bar{I} and the quotient $Q_I := L_a^2(\mathbb{B}^m)/\bar{I}$ are essentially normal A -modules. Furthermore, Q_I and $L_{a,M}^2(\Omega_I)$ correspond to the same class in $K_1(\partial\Omega_I)$.*

Proof. We consider the exact sequence of A -modules,

$$0 \longrightarrow \ker R_{\Omega_I} \xrightarrow{\iota} L_a^2(\mathbb{B}^m) \xrightarrow{R} L_{a,M}^2(\Omega_I) \longrightarrow 0.$$

By Theorem 4.3, R is a surjective A -module map. According to Corollary 1.3, $L_{a,M}^2(\Omega_I)$ is an essentially normal A -module. We conclude from [16, Thm. 1] that the kernel $\ker R$ is an essentially normal A -module as $L_a^2(\mathbb{B}^m)$ is also essentially normal. By Theorem 6.3, the closure \bar{I} of I in $L_a^2(\mathbb{B}^m)$ agrees with the kernel $\ker R$. Hence, \bar{I} is an essentially normal A -module.

Now consider the following exact sequence

$$0 \longrightarrow \bar{I} \longrightarrow L_a^2(\mathbb{B}^m) \longrightarrow Q_I \longrightarrow 0.$$

By [17, Thm. 1], the essentially normality of the ideal \bar{I} implies that the quotient module $Q_I = L_a^2(\mathbb{B}^m)/\bar{I}$ is essentially normal.

As $\bar{I} = \ker R$, $R_{\Omega_I} : Q_I \rightarrow L_{a,M}^2(\Omega_I)$ is an isomorphism of essentially normal A -modules. Apply Proposition 4.4 to R_{Ω_I} . We conclude that Q_I and $L_{a,M}^2(\Omega_I)$ give rise to the same K -homology class in $K_1(\partial\Omega_I)$. \square

Remark 4.6. *Theorem 4.5 states that Q_I and $L_{a,M}^2(\Omega_I)$ are equivalent K -homology classes. As is suggested in [4], a sharper estimate of Sobolev norms proves that $L_{a,s}^2(\Omega_I)$ is a Schatten- p class module for $p > k$. Our proof of Theorem 4.5 does not show that the ideal \bar{I} or the quotient module Q_I is a Schatten- p class module, though we expect that they are and Q_I is equivalent to $L_{a,M}^2$ as Schatten- p class modules.*

Remark 4.7. *In Theorem 4.5, we proved that the kernel $\ker R$ is an essentially normal A -module, when Z_I is the zero variety of functions f_1, \dots, f_M that are holomorphic on a neighborhood of the closed ball $\overline{\mathbb{B}}^m$ and satisfy Assumption 1.1. This is a variant of the “Geometric Arveson-Douglas Conjecture” proposed by Engliš and Eschmeier [20]. Engliš*

and Eschmeier [20] proved that the kernel $\ker R$ is an essentially normal A -module when Z_I is a homogeneous variety with the only singularity at the origin $O \in \mathbb{C}^m$. This result and our Theorem 4.5 are independent supporting evidences for the “Geometric Arveson-Douglas Conjecture.” The two results use different methods. The main tool in [20] is the Boutet de Monvel-Guillemin theory of generalized Toeplitz operators, while the main tool in this paper is the Baum-Douglas-Taylor theory of relative K -homology for manifolds with boundaries.

In [18], the first author and Wang proved that when I is a principal ideal of $A = \mathbb{C}[z_1, \dots, z_m]$, the quotient Hilbert module Q_I is essentially normal. Let p be a generator of I . The zero set of p is a hypersurface Z_I of \mathbb{C}^m . Assumption 1.1 in this case requires that the 1-form dp is everywhere nonzero on $\partial\Omega_I$ and Z_I intersects with the sphere $\partial\mathbb{B}^m$ transversely.

Corollary 4.8. *For $m \geq 3$, when I is generated by $p \in A$ and satisfies Assumption 1.1, the K -homology class of Q_I is the fundamental class of $\partial\Omega_I$.*

As a special example of Corollary 4.8, we consider the following polynomial

$$p_k(z_1, \dots, z_5) = z_1^2 + z_2^2 + z_3^2 + z_4^3 + z_5^{6k-1} \in \mathbb{C}[z_1, \dots, z_5], \quad k \geq 1.$$

The zero variety Z_{p_k} of p_k has an isolated singularity at the origin, and when $\epsilon > 0$ is sufficiently small, Z_{p_k} intersects with the sphere $\mathbb{S}_\epsilon^9 = \partial\mathbb{B}_\epsilon^5 = \overline{\mathbb{B}_\epsilon^5} \setminus \mathbb{B}_\epsilon^5$ transversely [11], [30], where \mathbb{B}_ϵ^5 is the open ball of radius ϵ around the origin. Hence the conditions of Corollary 4.8 are satisfied on \mathbb{B}_ϵ^5 . We conclude that Q_{I_k} gives the fundamental class of the boundary $\partial\Omega_{I_k}^\epsilon = Z_{p_k} \cap \partial\mathbb{B}_\epsilon^5$. The boundary $\partial\Omega_{I_k}^\epsilon$ is a topological 7-sphere \mathbb{S}^7 . When $k = 1, \dots, 28$, the differentiable structures on $Z_{p_k} \cap \partial\mathbb{B}_\epsilon^5$ give all the different differentiable structures on \mathbb{S}^7 . Corollary 4.8 offers a possibility to use operator algebra tools to study differentiable topology on \mathbb{S}^7 . We plan to come back to this question in the near future.

5. CONCLUDING REMARKS

We end this article with a few remarks.

5.1. Assumption 1.1 on complete intersection. Our results in this article rely crucially on Assumption 1.1 that the ideal I is generated by $p_1, \dots, p_M \in \mathbb{C}[z_1, \dots, z_m]$ with $M \leq m - 2$ such that the Jacobian matrix $(\partial p_i / \partial z_j)$ is of maximal rank on the boundary $\partial\Omega_I = Z_I \cap \partial\mathbb{B}^m$ and the zero variety Z_i intersects $\partial\mathbb{B}^m$ transversely.

Using the concept of depth [19, Sec.18.5] in algebraic geometry, we can easily show that Assumption 1.1 implies that the ideal I is radical. Let Σ_I be the set of singular sets in Ω_I . Assumption 1.1 also implies that Σ_I is a finite discrete set having no intersection with the boundary $\partial\Omega_I$. Furthermore, Assumption 1.1 implies that A/I is a complete

intersection ring [19, Sec.18.5], which assures that [27, Ch.3, Ex.3.5] any analytic function on $\Omega_I - \Sigma_I$ has an extension to Ω_I . This is the key in our proof that the restriction map $R : L_a^2(\mathbb{B}^m) \rightarrow L_{a,M}^2(\Omega_I)$ is surjective.

As is explained in Proposition 4.8, there are many examples satisfying Assumption 1.1. On the other hand, there are also many interesting examples on which Assumption 1.1 fails to hold. Let $\langle z_1, z_2 \rangle$ (and $\langle z_3, z_4 \rangle$) be ideals of $\mathbb{C}[z_1, \dots, z_4]$ generated by z_1, z_2 (and z_3, z_4). Consider the ideal I of $\mathbb{C}[z_1, \dots, z_4]$ as the product of $\langle z_1, z_2 \rangle$ and $\langle z_3, z_4 \rangle$. I is generated by $z_1 z_3, z_1 z_4, z_2 z_3, z_2 z_4$. If Assumption 1.1 holds for I , Z_I has dimension 0. But the zero variety of I has dimension 2. This shows that I fails to satisfy Assumption 1.1. In general, the ideal I fails to satisfy the Cohen-Macaulay condition [19, Sec.18.2], and therefore there are analytic functions on $\Omega_I - \Sigma_I$ that cannot be extended to Ω_I . Such a failure suggests that the restriction map $R : L_a^2(\mathbb{B}^m) \rightarrow L_{a,M}^2(\Omega_I)$ cannot be surjective for the ideal $I = \langle z_1, z_2 \rangle \langle z_3, z_4 \rangle$. On the other hand, in this case the range of the restriction map R still has a finite codimension, which is sufficient for us to conclude Theorem 4.5. This suggests to generalize our results in this article to include examples like $I = \langle z_1, z_2 \rangle \langle z_3, z_4 \rangle$. We will come back to this in the near future.

5.2. Non-radical ideals. In this article, we have considered only radical ideals. Chen, the first author, Keshari, and Xu take up the simplest non-radical cases, the ideal I_α generated by the monomial $\underline{z}^\alpha = z_1^{\alpha_1} \cdots z_m^{\alpha_m}$ for non-negative integers $(\alpha_1, \dots, \alpha_m)$. They show that I_α defines a sequence of zero sets $Z_k \subseteq Z_{k-1} \subseteq \cdots \subseteq Z_2 \subseteq Z_1$, where Z_i is a union of hyperplanes intersecting at the origin such that a hyperplane for Z_i contains a hyperplane for Z_{i+1} or is otherwise distinct from it.

The ideas in this paper can be applied to study non-radical ideals. When I is not radical, the kernel of the restriction map $R : L_a^2(\mathbb{B}^m) \rightarrow L_{a,M}^2(\Omega_I)$ is not the closure of I . However, by taking the zero variety Z_I in the sense of schemes in algebraic geometry, we expect to enrich the space $L_{a,M}^2(\Omega_I)$ to a bigger Hilbert space $\mathcal{L}_{a,M}^2(\Omega_I)$. With the bigger space $\mathcal{L}_{a,M}^2(\Omega_I)$, we expect to show that the restriction map \mathcal{R} is surjective or has a finite codimensional range and the kernel $\ker(R)$ is the closure of I . With this modification, we can use the methods developed in this article to study the essentially normal property of \bar{T} and Q_I .

The following example might be useful in understanding this phenomenon. Let $m = 2$, and I, J be the principal ideals generated by z_1 and z_1^2 , respectively. Then the zero variety of both I and J is $Z_I = Z_J = \{z_1 = 0\}$. And $\Omega_I = \Omega_J$ is $Z_I \cap \mathbb{B}^2$. The kernel $\ker R$ of the usual restriction map $R_I : L_a^2(\mathbb{B}^2) \rightarrow L_{a,1}^2(\Omega_I)$ is the closure of I in $L_a^2(\mathbb{B}^2)$. Let $\mathcal{L}_{a,1}^2(\Omega_J)$ be $L_{a,1}^2(\Omega_I) \oplus L_{a,2}^2(\Omega_I)$. Define the restriction map $R_J : L_a^2(\mathbb{B}^2) \rightarrow \mathcal{L}_{a,1}^2(\Omega_J)$ to be

$$R_J(f) = f|_{z_1=0} \oplus \frac{\partial f}{\partial z_1}|_{z_1=0}.$$

It is straight forward to check that the kernel $\ker R_J$ of the map R_J is the closure of J in $L_a^2(\mathbb{B}^2)$, and R_J is surjective. Therefore, the results of this paper extend to show that the closure \overline{J} and the quotient $L_a^2(\mathbb{B}^2)/\overline{J}$ are essentially normal A -modules (see also [25]).

6. APPENDIX: THE CLOSURE OF THE IDEAL I IN $L_a^2(\mathbb{B}^m)$

by R. Douglas, X. Tang, K. Wang², and G. Yu

In this appendix, we study the closure of the ideal I in $L_a^2(\mathbb{B}^m)$. We prove that under Assumption 1.1 if $f \in L_a^2(\mathbb{B}^m)$ vanishes on Ω_I , then f is contained in the closure \overline{I} of I in $L_a^2(\mathbb{B}^m)$. Related results can be found in [1] and [33].

Let $\mathcal{O}(\overline{\mathbb{B}^m})$ be the algebra of holomorphic functions on the closed ball $\overline{\mathbb{B}^m}$. $\mathcal{O}(\overline{\mathbb{B}^m})$ with the natural topology is a (topological) unital commutative noetherian ring [23] satisfying the Hilbert nullstellensatz, i.e. every ideal of $\mathcal{O}(\overline{\mathbb{B}^m})$ is either dense, or contained in a maximal ideal \mathcal{M} in $\mathcal{O}(\overline{\mathbb{B}^m})$ such that the \mathcal{M} -adic topology is weaker than the topology on $\mathcal{O}(\overline{\mathbb{B}^m})$.

Lemma 6.1. *Let I be a prime ideal of $\mathcal{O}(\overline{\mathbb{B}^m})$ and $f \in \mathcal{O}(\overline{\mathbb{B}^m})$. Denote Z_I to be the zero variety of the ideal I . If there is a point $z_0 \in Z_I \cap \mathbb{B}^m$ and an open set U of z_0 in \mathbb{B}^m , such that f belongs to the ideal $I\mathcal{O}(U)$ generated by I in $\mathcal{O}(U)$, the ring of holomorphic functions on U , then $f \in I$.*

Proof. Let M_{z_0} be the maximal ideal of $\mathcal{O}(\overline{\mathbb{B}^m})$ generated by analytic functions in $\mathcal{O}(\overline{\mathbb{B}^m})$ vanishing at z_0 . Consider the M_{z_0} -adic completion of I in $\mathcal{O}(\overline{\mathbb{B}^m})$. As $z_0 \in Z_I$, $I \subseteq M_{z_0}$ and $I + M_{z_0} = M_{z_0}$. By Krull's theorem [14, Sec. 2.1], the M_{z_0} -adic closure of I in $\mathcal{O}(\overline{\mathbb{B}^m})$ is I . By definition, the M_{z_0} -adic closure of $\mathcal{O}(\overline{\mathbb{B}^m})$ is

$$\bigcap_{j \geq 1} [I + M_{z_0}^j].$$

Consider the function f . We prove that f is in $I + M_{z_0}^j$ for every $j \geq 1$. On U , as f belongs to the ideal $I\mathcal{O}(U)$, there are $p_1, \dots, p_l \in I$, and $h_1, \dots, h_l \in \mathcal{O}(U)$ such that

$$f = \sum_{i=1}^l p_i h_i.$$

Choose g_1, \dots, g_l in the polynomial ring $A = \mathbb{C}[z_1, \dots, z_m]$ such that $h_i - g_i$ vanishes at z_0 of multiplicity j . Observe that

$$f - \sum_{i=1}^l p_i g_i = \sum_{i=1}^l p_i (h_i - g_i)$$

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vanishes at z_0 of multiplicity j . Hence, $f - \sum_{i=1}^l p_i g_i$ is contained in $M_{z_0}^j$. Therefore, we have proved that f belongs to $I + M_{z_0}^j$. Varying j through all natural numbers, we conclude that f belongs to $I = \bigcap_{j \geq 1} [I + M_{z_0}^j]$. \square

Lemma 6.2. *Under Assumption 1.1, if $f \in \mathcal{O}(\overline{\mathbb{B}}^m)$ vanishes on Ω_I , then f belongs to the closure $\langle I \rangle$ of I in $\mathcal{O}(\overline{\mathbb{B}}^m)$, and therefore in the closure \overline{I} of I in $L_a^2(\mathbb{B}^m)$.*

Proof. We consider $\mathcal{O}(\overline{\mathbb{B}}^m)$ with the topology from $H^\infty(\mathbb{B}^m)$ and the ideal $I\mathcal{O}(\overline{\mathbb{B}}^m)$ in $\mathcal{O}(\overline{\mathbb{B}}^m)$. Let $I\mathcal{O}(\overline{\mathbb{B}}^m) = \bigcap_i^l I_i$ be the irredundant primary decomposition of $I\mathcal{O}(\overline{\mathbb{B}}^m)$. By [23, Theorem 1.3], the closure of $I\mathcal{O}(\overline{\mathbb{B}}^m)$ in $\mathcal{O}(\overline{\mathbb{B}}^m)$ is the intersection of those I_i that is contained in a closed maximal ideal M_{z_i} . It is sufficient to prove that f belongs to all those I_i that is contained in a closed maximal ideal M_{z_i} .

By Assumption 1.1, Z_I is smooth near the boundary $\partial\mathbb{B}^m$ and intersects with $\partial\mathbb{B}^m$ transversely. Hence, if there is a point $z_i \in Z_{I_i} \cap \partial\mathbb{B}^m$, z_i is a smooth point of Z_I , and therefore a smooth point of Z_{I_i} . We remark that as z_i is a smooth point of Z_I , there can not be a different $I_{i'}$ such that Z_{I_i} and $Z_{I_{i'}}$ intersect at z_i . Since Z_I intersects with the boundary $\partial\mathbb{B}^m$ transversely, Z_{I_i} must intersect with the boundary $\partial\mathbb{B}^m$ transversely at z_i too. So there must be a point \tilde{z}_i in $Z_{I_i} \cap \mathbb{B}^m$. Furthermore, on a sufficiently small neighborhood U_i of \tilde{z}_i in \mathbb{B}^m , f vanishes on $Z_{I_i} \cap U_i$, and therefore f is contained in the ideal $I_i\mathcal{O}(U_i)$ generated by I_i in $\mathcal{O}(U_i)$. Lemma 6.1 applies to f and I_i with U_i and \tilde{z}_i . Therefore, f is contained in I_i for all i such that I_i is contained in a closed maximal ideal M_{z_i} . Hence, f is in the closure $\langle I \rangle$ of I in $\mathcal{O}(\overline{\mathbb{B}}^m)$, and therefore also in the closure \overline{I} in $L_a^2(\mathbb{B}^m)$. \square

Let R be the restriction operator $R : L_a^2(\mathbb{B}) \rightarrow L_{a,M}^2(\Omega_I)$.

Theorem 6.3. *Under Assumption 1.1, the closure \overline{I} of the ideal I in $L_a^2(\mathbb{B}^m)$ is equal to $\ker R$.*

Proof. We observe that if f belongs to $\mathcal{O}(\overline{\mathbb{B}}^m) \cap \ker R$, then Lemma 6.2 implies that f belongs to the closure \overline{I} of I in $L_a^2(\mathbb{B}^m)$. In the following, we show that every function in $\ker R$ can be approximated arbitrarily closely by functions in $\mathcal{O}(\overline{\mathbb{B}}^m) \cap \ker R$.

Let f be a function in the kernel $\ker R$. For $0 < r < 1$, define f_r to be $f_r(z) := f(rz)$. f_r is a holomorphic function on the (open) ball $\mathbb{B}_{1/r}^m$ of radius $1/r$ in \mathbb{C}^m . Furthermore, we have the following properties of f_r on the unit ball \mathbb{B}^m .

(1)

$$\lim_{r \rightarrow 1} f_r(z) = f(z), \quad \forall z \in \mathbb{B}^m.$$

(2)

$$\lim_{r \rightarrow 1} \|f_r\|_{L_a^2(\mathbb{B}^m)} = \|f\|_{L_a^2(\mathbb{B}^m)}.$$

From the above two properties, we conclude that f_r converges to f in $L_a^2(\mathbb{B}^m)$ as $r \rightarrow 1$.

Fix $\epsilon > 0$. Since f_r converges to f in $L_a^2(\mathbb{B}^m)$, there exists a number r_1 with $0 < r_1 < 1$ such that $\|f - f_{r_1}\|_{L_a^2(\mathbb{B}^m)} < \epsilon$. As $R(f) = 0$, $R(f - f_{r_1}) = -R(f_{r_1}) \in L_{a,M}^2(\Omega_I)$. Furthermore, as $R : L_a^2(\mathbb{B}^m) \rightarrow L_{a,M}^2(\Omega_I)$ is bounded,

$$\|R(f_{r_1})\|_{L_{a,M}^2(\Omega_I)} = \|R(f - f_{r_1})\|_{L_{a,M}^2(\Omega_I)} \leq \|R\| \cdot \|f - f_{r_1}\|_{L_a^2(\mathbb{B}^m)} = \|R\|\epsilon.$$

For $0 < s < 1$, consider the ball $\mathbb{B}_{1/s}^m \subset \mathbb{C}^m$ of radius $1/s$ with the boundary $\partial\mathbb{B}_{1/s}^m = \mathbb{S}_{1/s}^{2m-1}$. Let Ω_I^s be the intersection of $\mathbb{B}_{1/s}^m$ with Z_I . Assumption 1.1 implies that there is a number S with $0 < S < 1$ such that the similar properties as in Assumption 1.1 also hold on the Ω_I^s and $\partial\Omega_I^s$ for all s with $S < s < 1$. Let $L_a^2(\mathbb{B}_{1/s}^m)$ and $L_{a,M}^2(\Omega_I^s)$ be the corresponding (weighted) Bergman spaces. Then Theorem 4.3 can be naturally generalized to produce an extension operator

$$E_s : L_{a,M}^2(\Omega_I^s) \longrightarrow L_a^2(\mathbb{B}_{1/s}^m),$$

and a restriction operator

$$R_s : L_a^2(\mathbb{B}_{1/s}^m) \longrightarrow L_{a,M}^2(\Omega_I^s),$$

such that $R_s E_s = Id$. Furthermore, following the estimates in [7, Lemma 3.10 and Theorem 4.1], we can obtain that there is a number S' with $0 < S' < 1$ and $M > 0$ such that $\|E_s\| < M$ for all s satisfying $S' < s < 1$.

When $r_1 < s < 1$, $R_s(f_{r_1})$ is well defined in $L_{a,M}^2(\Omega_I^s)$. As Z_I intersects with $\mathbb{S}_{1/s}^{2m-1}$ transversely, the defining function $\rho_s(z)$ for Ω_I^s is continuous with respect to s . When s goes to 1, $\|R_s(f_{r_1})\|_{L_{a,M}^2(\Omega_I^s)}$ converges to $\|R(f_{r_1})\|_{L_{a,M}^2(\Omega_I)}$. In particular, there is a number s_1 with $r_1 < s_1 < 1$, such that

$$\|R_{s_1}(f_{r_1})\|_{L_{a,M}^2(\Omega_I^{s_1})} \leq 2\|R(f_{r_1})\|_{L_{a,M}^2(\Omega_I)} \leq 2\|R\|\epsilon.$$

As the norm of E_s is uniformly bounded by M for $S < s < 1$, we will choose s_1 such that $\|E_{s_1}\| < M$. Then $\|E_{s_1} R_{s_1}(f_{r_1})\|_{L_a^2(\mathbb{B}_{1/s_1}^m)}$ is bounded by $2M\|R\|\epsilon$.

Observe that both f_{r_1} and $E_{s_1} R_{s_1}(f_{r_1})$ are holomorphic on $\overline{\mathbb{B}}^m$. Define $F_\epsilon \in \mathcal{O}(\overline{\mathbb{B}}^m)$ by $F_\epsilon(z) := f_{r_1}(z) - E_{s_1} R_{s_1}(f_{r_1})(z)$. Then

$$R(F_\epsilon) = R(f_{r_1}) - R(E_{s_1} R_{s_1}(f_{r_1})) = R(f_{r_1}) - R(f_{r_1}) = 0.$$

Hence, F_ϵ is inside $\ker R$, and therefore in $\mathcal{O}(\overline{\mathbb{B}}^m) \cap \ker R$. We have the following estimate of the norm $\|f - F_\epsilon\|_{L_a^2(\mathbb{B}^m)}$.

$$\begin{aligned} \|f - F_\epsilon\|_{L_a^2(\mathbb{B}^m)} &= \|f - f_{r_1} + E_{s_1} R_{s_1}(f_{r_1})\|_{L_a^2(\mathbb{B}^m)} \leq \|f - f_{r_1}\|_{L_a^2(\mathbb{B}^m)} + \\ &\quad + \|E_{s_1} R_{s_1}(f_{r_1})\|_{L_a^2(\mathbb{B}^m)} \leq \epsilon + \|E_{s_1} R_{s_1}(f_{r_1})\|_{L_a^2(\mathbb{B}_{1/s_1}^m)} \leq (2M\|R\| + 1)\epsilon. \end{aligned}$$

Hence we conclude that f can be approximated arbitrarily closely by functions in $\mathcal{O}(\overline{\mathbb{B}}^m) \cap \ker R$, and therefore f is in the closure \overline{I} of I in $L_a^2(\mathbb{B}^m)$. This shows that $\ker R$ is contained inside \overline{I} .

Finally, as \bar{I} is naturally contained in $\ker R$, we obtain that $\bar{I} = \ker R$.

□

The result in Theorem 6.3 was stated in a more general context without Assumption 1.1 in [34, Theorem. 4.1 and Remark 4.4]. But Mihai Putinar and Kunyu Guo explained to us that there are mistakes in the proofs. We refer the readers to [1] and [33] for more results along this direction. In general it is an open question if the closure \bar{I} of the ideal I in $L_a^2(\mathbb{B}^m)$ is equal to the kernel of R when I is radical.

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